SYMmetryplus

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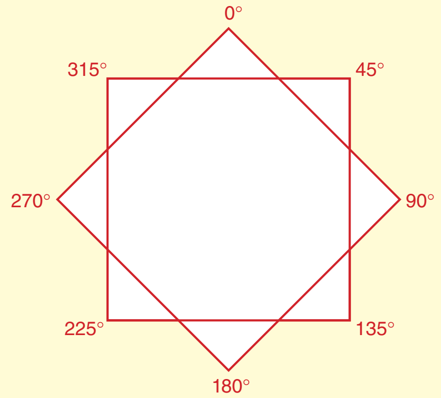
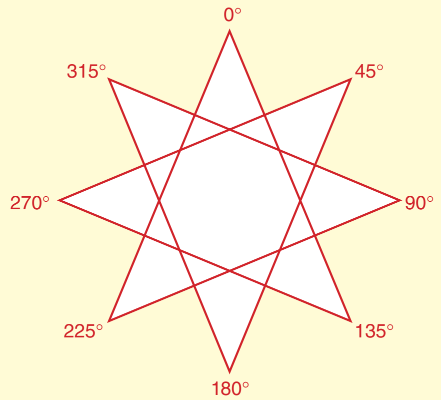
EDITORIAL

This will be my last issue of SYMmetry*plus*. My successor will be Michael Moon. I wish him all the best for future issues. Please send him all the exciting contributions you were about to send me. Michael's address is:

2 Telford Terrace, Albemarle Road, York YO24 1DQ
It's been fun doing this for the past few years, going back to the original SYMmetry, but it's time to hand on.

Martin Perkins.

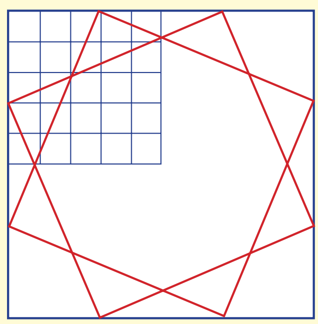




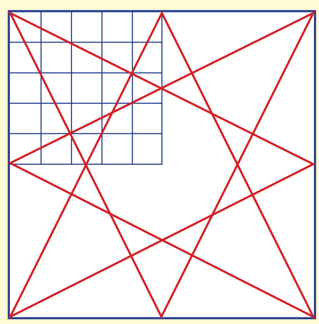
ISLAMIC ART

We have mentioned before the excellent series of books produced by Tarquin on Geometric Patterns, all written by Robert Field. The latest one deals with Geometric Patterns from the Islamic world. Representation of the human or animal form was prohibited as a way of decorating mosques and other religious buildings early in the history of Islam, so artists concentrated on using formal geometric patterns, becoming very skilled as the patterns became more and more complex. The book starts with a quotation from the fourteenth century Arabic philosopher Ibn Khuldun: "The mind that constantly applies itself to geometry is not likely to fall into error."

The illustrations come from Spain to India, from Cairo to Cardiff. As well as the photographs there are, of course, the diagrams to show you how to construct the patterns for yourselves.



The illustrations shown here show different ways of producing eight pointed stars with a central octagon (not always regular). This is a very common motif in Islamic patterns. As always, the book is thoroughly enjoyable to read and well worth buying at £2.95.



[*Geometric Patterns from Islamic Art & Architecture,* by Robert Field, publ. Tarquin: ISBN 0 899618 22 8]

MLP

TWO PROBLEMS WITH 'NICE' SOLUTIONS

Neville Reed wrote some time ago with two problems which had what he called 'nice' solutions.

Consider a triangle with sides 13, 14 and 15 units. What is its area? We can solve this using Hero's formula.

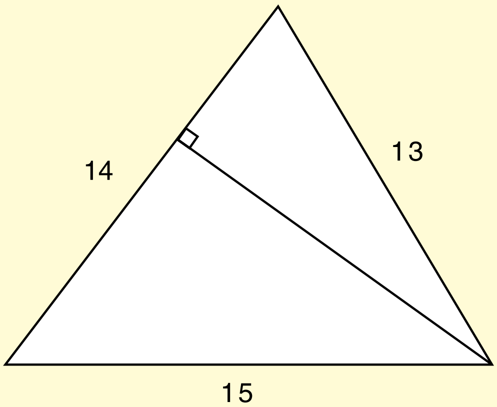
If the sides of the triangle are a, b and c, then the area, $\Delta = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.
s is the <u>s</u>emi-perimeter.

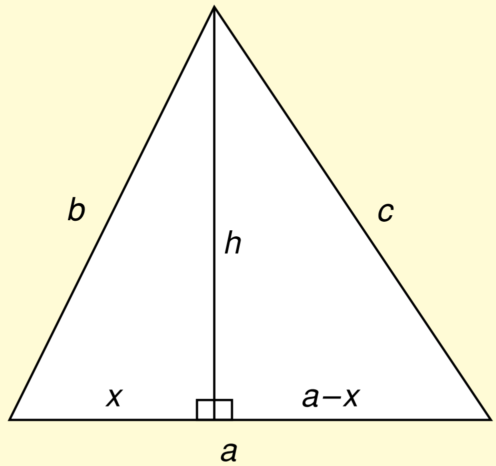
Here we have $a = 13$, $b = 14$, $c = 15$ so $s = \frac{1}{2}\,(13+14+15) = 21$; $s-a = 8$, $s-b = 7$, $s-c = 6$

$$\therefore \quad \Delta = \sqrt{21\times8\times7\times6} = \sqrt{7\times3\times4\times2\times7\times3\times2} = \sqrt{7^2\times3^2\times4^2} = 7\times3\times4 = 84.$$

(Who needs a calculator?)

As a whole number this is certainly 'nice'. Incidentally, because $84 = 14\times6 = \frac{1}{2}\times14\times12$ we can see that one of the heights of the triangle is 12, also a 'nice' result. The coincidence of 12, 13, 14 and 15 in the same diagram is also rather pleasing. Perhaps you can find some more 'nice' triangles.





We ought not to use Hero's formula without showing a proof. Hero, or Heron, was an Alexandrian Greek who lived probably in the last first century AD, some 300 years before Hypatia about whom we can read on page six. He was as much an engineer as a mathematician, writing about water and steam power – it is believed that he invented machines for moving scenery in theatres to create special effects such as the gods descending, and taking an interest also in surveying, mensuration and optics – he showed that the angle of reflection is equal to the angle of incidence. This famous result is probably due to Archimedes, but Hero's proof is the earliest we have. Referring to the right hand diagram:

$\Delta = \frac{1}{2}ah$ and from Pythagoras' Theorem: $h^2 = b^2 - x^2 = c^2 - (a-x)^2 = c^2 - a^2 + 2ax - x^2$

From this we have $b^2 = c^2 - a^2 + 2ax$ and hence $x = \dfrac{a^2+b^2-c^2}{2a}$.

$$\therefore \quad h^2 = b^2 - x^2 = \frac{4a^2b^2 - (a^2+b^2-c^2)^2}{4a} = \frac{(2ab - a^2 - b^2 - c^2)\,(2ab + a^2 + b^2 - c^2)}{4a^2}$$

$$= \frac{[c^2-(a-b)^2]\,[(a+b)^2-c^2]}{4a^2} = \frac{(c-a+b)\,(c+a-b)\,(a+b+c)\,(a+b-c)}{4a^2}$$

Now, since $2a = a + b + c$, $c + b - a = 2s - 2a$, etc

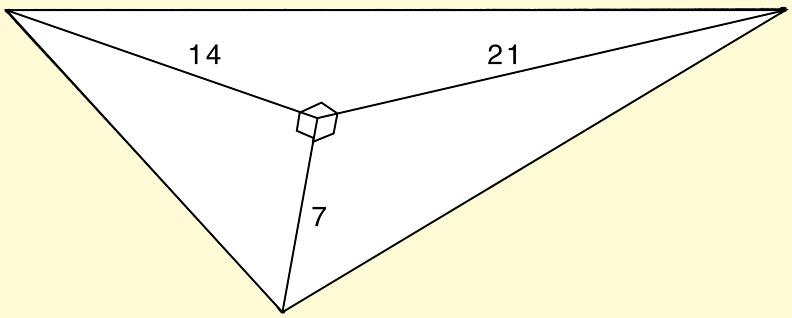
$$\therefore \quad 4a^2h^2 = (2s-2a)\,(2s-2b)\,(2s)\,(2s-2c) = 16s\,(s-a)\,(s-b)\,(s-c)$$

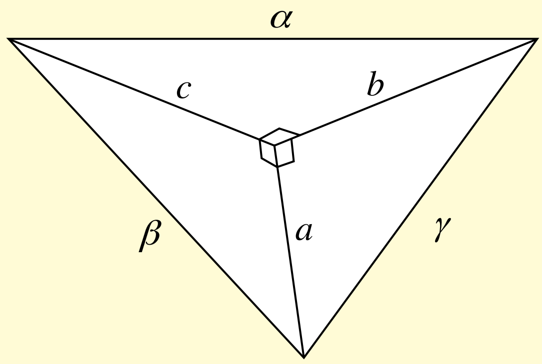
Hence $\Delta^2 = (\frac{1}{2}ah)^2 = \frac{1}{4}a^2h^2 = s(s-a)\,(s-b)\,(s-c)$

i.e., as required, $\quad \Delta = \sqrt{s(s-a)\,(s-b)\,(s-c)}$.

In the seventh century the Indian mathematician Brahmagupta produced a similar formula for the area of a cyclic quadrilateral (i.e. a quadrilateral whose vertices lie on a circle) as $\sqrt{s(s-a)\,(s-b)\,(s-c)\,(s-d)}$, where a, b, c and d are the lengths of the sides, and s is the semiperimeter, but that's another story . . .

Neville Read's second problem concerns a rather eccentric tent. It is held up by three poles of lengths 7, 14 and 21 feet which are mutually perpendicular at the top. How high is the tent?





The demonstration uses a sort of three-dimensional version of Pythagoras' Theorem: with three right angles at the apex the tent, which is of course a tetrahedron which looks like the corner cut off a cuboid, is a sort of solid right-angled triangle. In two dimensions Pythagoras' Theorem deals with the squares of the boundaries, i.e. the lengths of the sides of the triangle; in three dimensions we deal with the squares of the boundaries, i.e. the areas of the four triangle faces.

If the areas of the right-angled triangles are A_1, A_2 and A_3 and that of the fourth triangle (the floor of the tent) is A,

then $$A^2 = (A_1)^2 + (A_2)^2 + (A_3)^2 \qquad (*)$$

In this case we have
$$A^2 = (\tfrac{1}{2} \times 7 \times 21)^2 + (\tfrac{1}{2} \times 21 \times 14)^2 + (\tfrac{1}{2} \times 14 \times 7)^2$$
$$= (\tfrac{1}{2} \times 7 \times 7)^2 \times (3^2 + 6^2 + 2^2)$$
$$= (\tfrac{1}{2} \times 49)^2 \times 49$$

Hence $$A = \tfrac{1}{2} \times 49 \times 7$$

If we now use the formula for the volume of a tetrahedron, $V = \frac{1}{3} \times \textit{(base area)} \times \textit{(height)}$, in two ways we have

$$\tfrac{1}{3} \times A \times h = \tfrac{1}{3} \times (\tfrac{1}{2} \times 7 \times 14) \times 21 = \tfrac{1}{3} \times (\tfrac{1}{2} \times 49 \times 7) \text{ x } h$$

Hence $h = 6$. Clearly this qualifies as a 'nice' result.

We must, of course, justify the relation (*). We make use of Hero's formula.

Suppose that the mutually perpendicular edges are of lengths a, b and c. Let the hypotenuses of the right angled triangles be α, β and γ, so that

$$\alpha^2 = b^2 + c^2,\ \beta^2 = c^2 + a^2,\ \gamma^2 = a^2 + b^2.$$

The areas of the triangles are $A_1 = \frac{1}{2}bc$, $A_2 = \frac{1}{2}ca$. $A_3 = \frac{1}{2}ab$.

Using Hero's formula we have $A = \sqrt{\sigma(\sigma-\alpha)(\sigma-\beta)(\sigma-\gamma)}$ *where* $\sigma = \dfrac{\alpha+\beta+\gamma}{2}$

Hence
$$A^2 = \left(\frac{\alpha+\beta+\gamma}{2}\right)\left(\frac{\beta+\gamma-\alpha}{2}\right)\left(\frac{\gamma+\alpha-\beta}{2}\right)\left(\frac{\alpha+\beta-\gamma}{2}\right)$$

i.e.
$$A^2 = \left(\frac{[\beta+\gamma]+\alpha}{2}\right)\left(\frac{[\beta+\gamma]-\alpha}{2}\right)\left(\frac{\alpha+[\gamma-\beta]}{2}\right)\left(\frac{\alpha-[\gamma-\beta]}{2}\right)$$
$$= \left(\frac{[\beta+\gamma]^2-\alpha^2}{4}\right)\left(\frac{\alpha^2-[\gamma-\beta]^2}{4}\right) = \left(\frac{\beta^2+\gamma^2-\alpha^2+2\beta\gamma}{4}\right)\left(\frac{\alpha^2-\beta^2-\gamma^2+2\beta\gamma}{4}\right)$$

Substituting:
$$A^2 = \left(\frac{2\beta\gamma+2a^2}{4}\right)\left(\frac{2\beta\gamma-2a^2}{4}\right) = \frac{\beta^2\gamma^2-a^4}{4} = \frac{(c^2+a^2)(b^2+a^2)-a^4}{4} = \frac{b^2c^2+c^2a^2+a^2b^2}{4}$$

Thus we have our result: $$A^2 = (\tfrac{1}{2}bc)^2 + (\tfrac{1}{2}ca)^2 + (\tfrac{1}{2}ab)^2 = (A_1)^2 + (A_2)^2 + (A_3)^2.$$

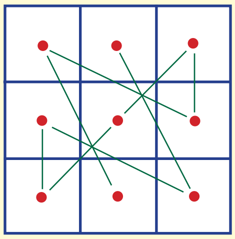
MLP

I was absolutely delighted to receive a set of notes from Canon Donald Eperson, who, over many years, has contributed so many wonderful ‘Puzzles, Pastimes & Problems’ to the MA journal *Mathematics in School*, on the subject of

MAGIC SQUARES

8	1	6
3	5	7
4	9	2

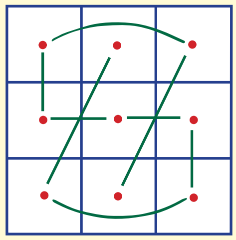
(1)



(2)

12	1	
	9	
		6

(3)



(4)

Magic Squares, in which the sum of the numbers in each row, column and diagonal is the same, provide a simple pastime that gives practice in simple arithmetic and also geometric pleasure.

Figure 1 shows an arrangement of the first nine positive integers as a magic square. It is essentially unique apart from reflections and rotations through multiples of 90°. Figure 2 shows that pattern formed by joining the centres of the cells by lines according to the ascending magnitude of the numbers. Other interesting patterns can be formed by joining the cells with even numbers, or the odd numbers, or the ‘arithmetic sequences’, 1, 2, 3; 1, 4, 7; 3, 6, 9; 7, 8, 9 in the same diagram. Figure 3 shows the beginning of another magic square which can be completed as it can be seen that the ‘magic total’ has to be $12 + 9 + 6 = 27$.

Figure 4 shows another pattern formed by cells in ascending order. This and the pattern in figure 2 are the only two patterns possible for a magic square with nine cells.

Note that the sum of the squares of the numbers in the top row of figure 1 is the same as the sum of the squares of the numbers in the bottom row. Is this true for the numbers in the two outside columns? And for figure 3?

$\frac{2}{3}$		
$\frac{1}{4}$		
$\frac{1}{3}$		$\frac{1}{6}$

(5)

3		2.8
3.2		
4		

(6)

	1	
X		
		2

(7)

47^2		23^2
	65^2	
89^2		79^2

(8)

You will need to hone your skills with fractions and decimals to complete the magic squares in figures 5 and 6.

Place any number you like in the central cell in figure 7 and complete the magic square; you will find that the cell marked *X* will contain the number 3. Can you explain why this is so?

In figure 8 the diagonal squares contain numbers that are perfect squares. Can you complete the magic square, and test that is is correct? Can you find another magic square which has perfect squares in its diagonal squares?

This last problem involves finding three perfect squares that form an arithmetic progression, i.e. three numbers *a, b, c* such that $a^2 + c^2 = 2b^2$ and then to find another such set with the same middle number, *b*. It can then be proved that *b* must be of the form $(x^2 + y^2)(z^2 + w^2)$, which can be written as $(xz \pm yw)^2 + (xw \mp yz)^2$.

If we set $(xz \pm yw) = p$ and $(xw \mp yz) = q$, taking the top sign or the bottom sign consistently, then we can have $a = (p + q)^2 - 2q^2$ and $c = (p + q)^2 - 2p^2$, which gives two sets of values for a and c. Figure 8 comes from setting $x = 1$, $y = 2$, $z = 2$, $w = 3$.

The central cell

In any magic 3×3 square with 'magic total' T the central square will always contain the number $\frac{T}{3}$.
If we add together the numbers down each diagonal and the central column, we have the following equations:

$a + m + z = T;\ \ b + m + y = T;\ \ c + m + x = T$

a	b	c
l	m	n
x	y	z

Adding these gives $\qquad a + b + c + 3m + x + y + z = 3T$

But the top and bottom rows show that $a + b + c = x + y + z = T$

Hence $\qquad 3m = T$, giving the required $m = \frac{T}{3}$.

With this in mind we can now construct a magic square to suit any requirements. We start by putting m in the central cell (figure i). Putting $m - x$ top left forces $m + x$ bottom right; likewise with $m - y$ and $m + y$ (figure ii). Knowing that the row and column totals all have to be $T = 3m$ allows us to complete the grid (figure iii).

	m	

i

m–x		m–y
	m	
m+y		m+x

ii

m–x	m+x+y	m–y
m+x–y	m	m–x+y
m+y	m–x–y	m+x

iii

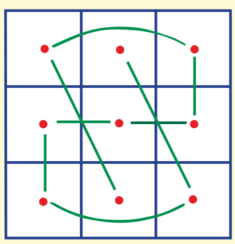
This square comes from setting $m = 9$, $x = 4$, $y = 3$.

5	16	6
10	9	8
12	2	13

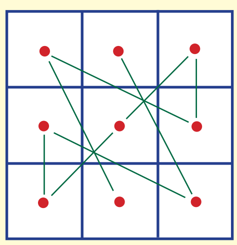
If we want to avoid negative numbers we must make sure that the smallest number, $m - x - y > 0$, i.e. $m > x + y$.
To avoid equal numbers occurring in two cells we find that we must ensure that obviously x and y must be different, and also that, for instance, $m - x + y \neq m - y$.
This gives the conditions $x \neq 2y$ and $y \neq 2x$.

If we assume that $x > y$ then we can look at the two possibilities $x < 2y$ and $x > 2y$; these are equivalent to the inequalities $x - y < y$ and $x - y > y$. We can then list the numbers in the cells in ascending order and join them up as we did right at the start. This shows that there are, indeed, only two possible patterns.

Case 1: $m - x - y,\ m - x,\ m - y,\ m - x + y,\ m,\ m + x - y,\ m + y,\ m + x,\ m + x + y$
Case 2: $m - x - y,\ m - x,\ m - x + y,\ m - y,\ m,\ m + y,\ m + x - y,\ m + n,\ m + x + y$



Case 1



Case 2

Donald B. Eperson

HYPATIA OF ALEXANDRIA

Hypatia was born about 370 AD in Alexandria in Egypt and is widely acknowledged as being the first female mathematician of note. Her father, who worked for the Museum of Alexandria, was Theon, a mathematician and astronomer. Hypatia was educated in both of these areas and showed early signs of being a genius, soon outstripping her teachers.

Hypatia studied with Plutarch the younger in Athens, and later in Italy, before returning home to Alexandria where she was invited to teach mathematics at the Neoplatonic school. Alexandria was a prestigious city, with wide streets, marble palaces, central heating, gardens, museums, a medical school and many universities. Its library had manuscripts on trigonometry, astronomy, geometry, hydraulics, navigation, circulation of the blood and also treatises on nervous diseases. It was a city with great promise for academics.

Hypatia was a popular teacher at the Neoplatonic school – one of her pupils, Synesius of Cyrene, described her as being "the illustrious and god-beloved philosopher". Said to be very beautiful, modest and intelligent, she had a melodic voice and a knack for explaining the most complex ideas in a clear and precise manner. She became the director of the institution in her early thirties and her home became an intellectual centre for discussions concerning the important scientific and philosophical matters of the day, drawing scholars from as far afield as Africa, Asia and Europe.

An idealised portrait of Hypatia

Hypatia wrote many items that served as texts for her students but unfortunately nothing has survived intact. Information on her writings comes from her devoted pupil, Synesius, who recorded much about her in his letters. Her most important work was a commentary on the *Arithmetica* of Diophantus, in which she included many alternative solutions and several new problems that were later incorporated into the books of Diophantus. She wrote another about the *Conics* of Apollonius, since she shared his fascination with conic sections. Other substantial commentaries by her were on Ptolemy's *Almagest* and Archimedes' *Measurement of the Circle.*

She was also interested in mechanics and practical technology and designed scientific instruments, including a planispheric astrolabe for use in measuring the altitudes of the stars and planets. She developed apparatus for distilling water, a graduated hydrometer for determining the specific gravity of liquids and a hydroscope for viewing objects far below the surface of water.

Hypatia was also passionate about politics and religion. At that time the accepted religion of the Roman Empire was Christianity, but the Christians of that era were violently opposed to the equality of women. Hypatia found their ways barbaric, since she was racially and religiously very tolerant and used to living in a city where not only was there great scope for education, but also in which women were treated as equals.

She followed the Neoplatonic ideal, which was to seek the revelation of the divide through knowledge. Her influence over Alexandria's Roman prefect, Orestes, was great and he was a Christian more by policy than conviction. This power made Cyril, the newly appointed bishop of Alexandria, very jealous. Cyril wanted Orestes to expel the 40,000 Jews from the city, but, advised by Hypatia, Orestes refused. Cyril took matters into his own hands and led a mob that burned Jewish synagogues and houses and stoned their inhabitants out of the city. Orestes protested and was himself attacked by Cyril's fanatical followers.

Word spread that Hypatia was the one causing the rift between Orestes and Cyril. This produced great anger and inflamed the Christians. During March, 415 AD, Hypatia met a grisly end. She was torn from her chariot by fanatical monks, stripped naked, dragged to the church and butchered. The flesh was scraped from her bones with sharp oyster shells and her bones were consigned to the flames.

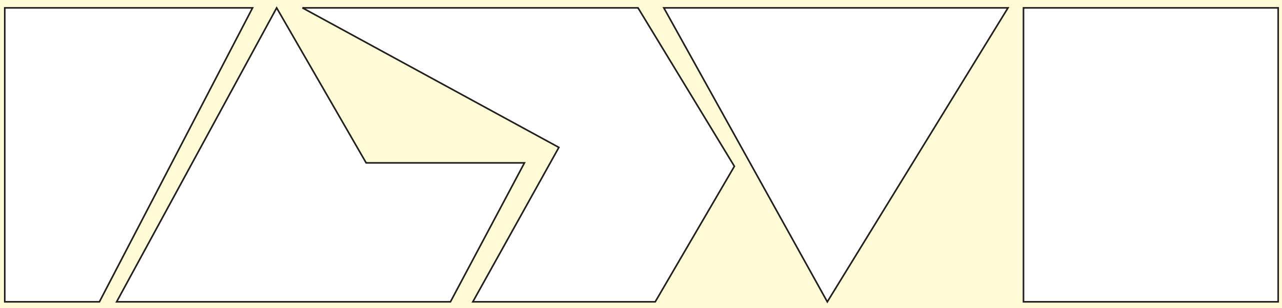
Cyril never condemned her death and no punishments were given for it. After her death Hypatia's pupils left the city, thus ending the era of Alexandria as the great mathematical centre of the world.

Jenny Ramsden

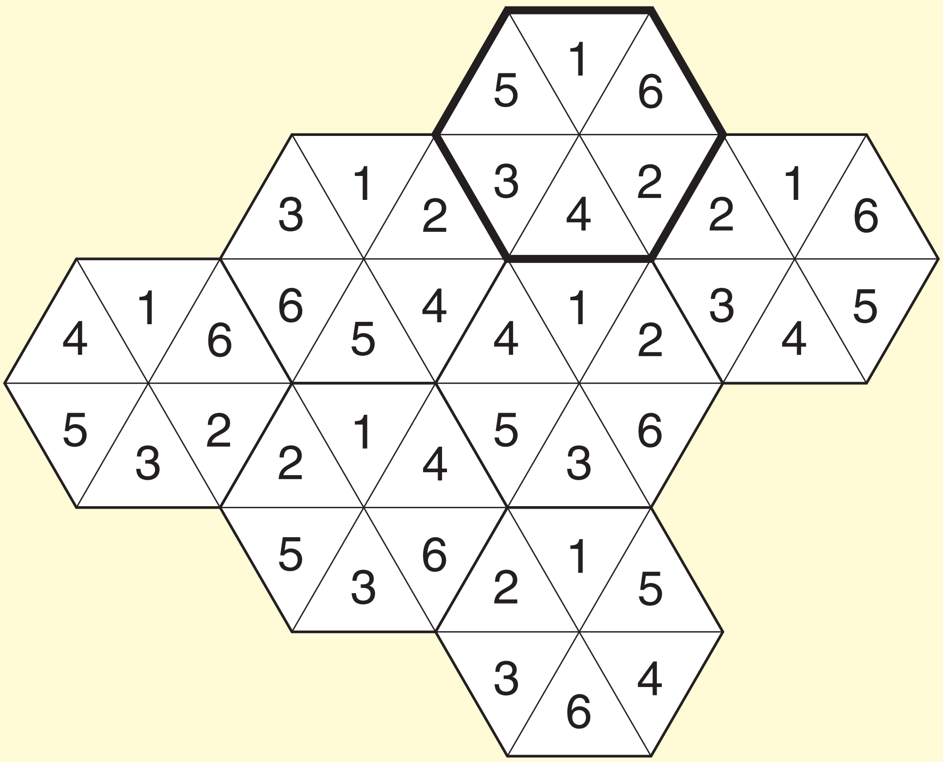
TWO HEXAGONS PROBLEMS

These two problems come from the "FUNMATHS ROADSHOW" pack produced by a team of teachers of Merseyside. Further details from:
Barry Grantham, Dept. of Mathematics, Liverpool Hope University College, Hope Park, Liverpool L16 9JD, or Ian Porteous, Dept. of Mathematical Sciences, The University of Liverpool, Liverpool, L69 3BX.

1. Reassemble these pieces to make a regular hexagon.

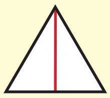


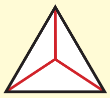
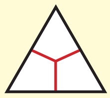
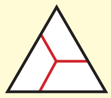
2. Copy and cut out these hexagons, then reassemble them on the pattern so that the numbers match where the hexagons share an edge. The top one, with the bold border, is in the right place already.



DEREK BALL'S PROBLEM

Into how many congruent pieces, p, can an equilateral triangle be divided?

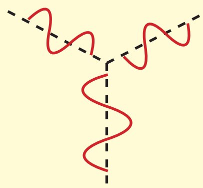
We can divide it into 2, like this: 

We can divide it into 3, like this:  or this:  or this: 

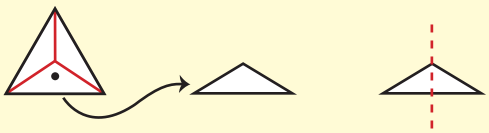
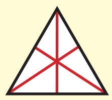
or even this: 

Our guide is symmetry.
The division into 2 uses one of the equivalent triangle's three axes of symmetry.
The division into 3 uses its centre of rotational symmetry, order 3. Our first three examples are made by sitting a 'trigon', i.e. a frame made of three rods fixed at 120° to each other, over the centre and turning it through different angles.

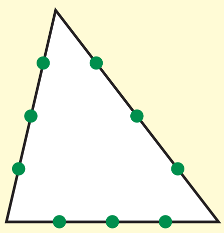
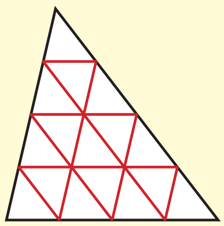
The fourth is made by taking one of these and deforming each of the 'radii' in the same way and in the same sense (to preserve the rotational symmetry).



If we find that the parts themselves are symmetrical, we can use **their** symmetries to divide the triangle further, e.g.

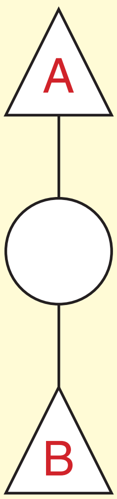
 giving us 6 pieces: 

Can we divide the equilateral triangle into 4? Yes. Indeed we can divide **any** triangle into 4 ($= 2^2$), 9 ($= 3^2$), 16 ($=4^2$) or any square number, n^2, of congruent pieces. We divide each side into n equal parts and join the divisions with lines parallel to the sides of the triangle:

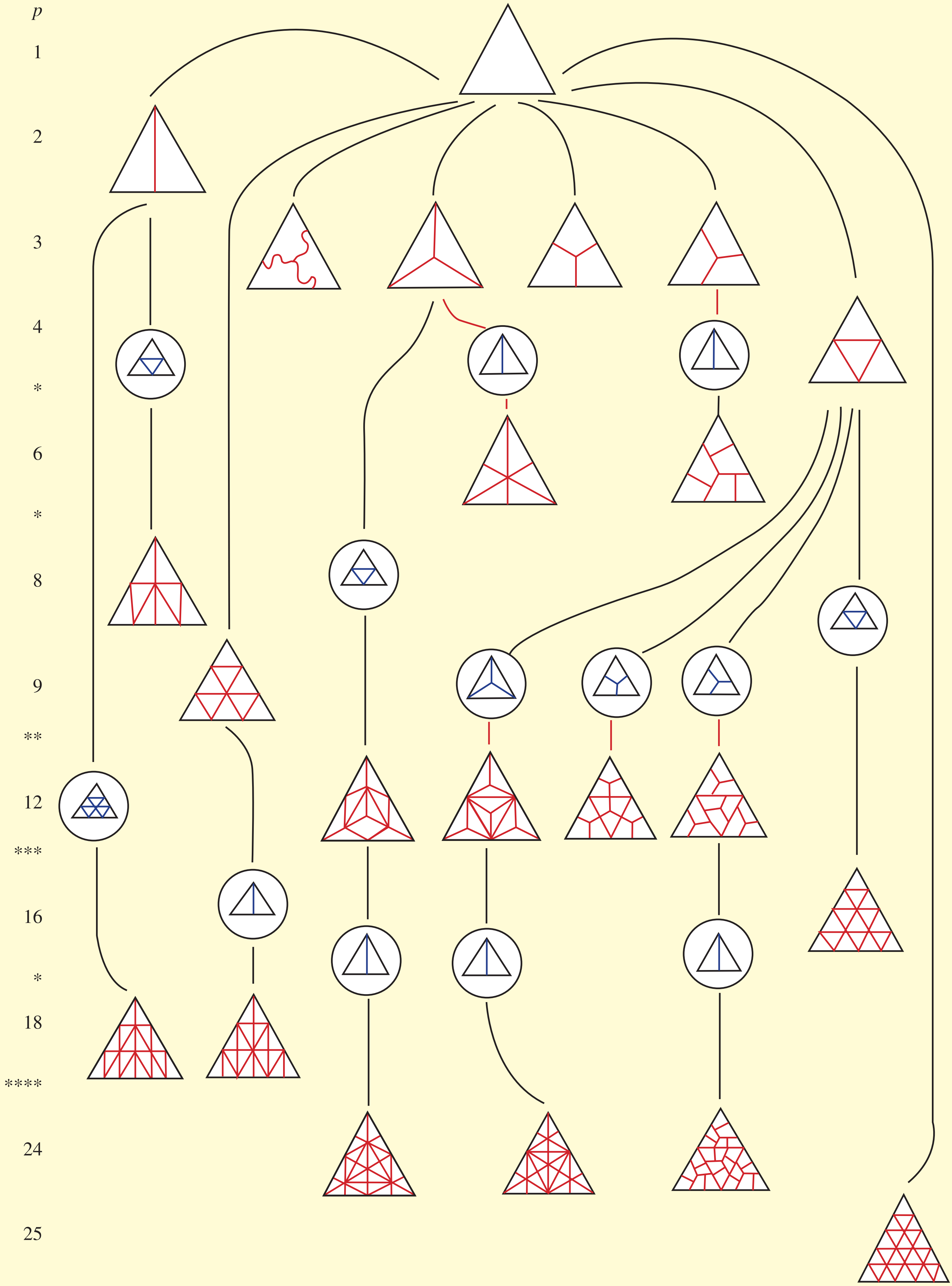
 

The chart on the next page shows the result of carrying out all these procedures as far as $p = 25$.

Key



What to do to each piece of *A* to produce *B*.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6
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8
9
**
12
***
16
*
18
****
24
25

There are several features to notice:

1. (Something we have already found) In some cases we can dissect the triangle in different ways to give the same number of pieces.

2. When there are such alternatives, we can often turn one into the other by using the symmetry of an internal polygon. Look, for example, at the two left hand dissections into 12. Find the regular hexagon within each and mentally give one $\frac{1}{6}$ of a turn.
More common examples are the bisected triangle, which can, of course, be produced on three ways, and the rectangles formed from the halves, which can be reflected in a symmetry axis. (See, for instance, the diagrams for $p = 18$.)

3. Some values are absent. List them and try to find out what they have in common. It may help if you decide which values are **present**.

4. There are sometimes alternative ways of achieving the same result, i.e. paths like those shown but which have not been included in the diagram. See if you can find a couple.

5. Each path shows an *operation*. There are rules by which these can be combined. If you give each operation a symbol and state these rules you have created an *algebra*, but a rather special one since it describes dissections of the equilateral triangle. What would excite a mathematician would be to find that the same algebra also described something quite different; such a match is called an *isomorphism*.

Here are two rules you can check from the diagram:
(i) You can only split a triangle into 3 **before** a division into 2 (why?).
(ii) You can split each constituent triangle into a square number of parts at **any** stage (why?).

Now we come to **Derek Ball's Problem**. Derek is the co-editor of the journal *Mathematics Teaching*. In a recent article on congruent dissections of regular polygons, Derek made the point: it is clear what you **can** do – in our diagram we've done it for the equilateral triangle – but not what you **cannot** do.

What about the missing values of p: 5, 7, 11, 13, ...?

If we forgot about symmetry but were sufficiently ingenious, could we dissect the equilateral triangle into **5** congruent pieces?

There are certain practical operations which mathematics shows to be impossible. One example is the trisection of an angle using only a straight edge and compasses. I remember that when our maths teacher told us this a friend of mine took it as a challenge and spent every breaktime for a week trying to prove him wrong. Needless to say he didn't succeed! However ingenious he was he could not have done so. Might the same be true of our problem? I don't know, but here is a conjecture:

'No regular n-gon can be split into p congruent pieces where n and p are distinct odd primes'

I couldn't prove anything so general, so I thought I'd settle for the special case $n = 3$, $p = 5$ – our first trouble spot, the equilateral triangle split into 5. I couldn't handle so many pieces and tried instead the case $n = 5$, $p = 3$ – the regular pentagon split into 3. Well, my proof is so long and untidy that I'm sure you can do better. Work on the problem with friends – perhaps even over the Internet – and, if you make a breakthrough, print out the result for SYMmetry*plus!*

Paul Stephenson
The Magic Mathworks Travelling Circus

MORE π

Well, not really, of course. But according to a report in *The Times* at the end of September two Japanese scientists have extended our knowledge of π to more than two billion decimal places. Dr Yasumasa Kanada and Dr Daisuke Tajahashi of the University of Tokyo have used a Hitachi SR8000 computer to establish π to 206,158,430,000 places.

Dr Roger Webster of Sheffield University, who is this country's leading π enthusiast, to whom Dr Kanada reported his result, said that reading the new value at one digit a second would take some 6500 years; and if written out and published as a book it would be 22 times as high as the Eiffel Tower – that's even getting on for twice the height of Mount Fuji!

MLP

EGYPTIAN FRACTIONS (concluded)

In the last issue we looked at the problems encountered by the Egyptians in dealing with fractions, since, apart from the exception of 2/3, they used only unit fractions i.e. fractions with a numerator of 1. To remind you of the process we discussed for decomposing any fraction into unit fractions here is the example we suggested you should have a go at:

$f = \frac{5}{13}$. The largest unit fraction less than or equal to f is $\frac{1}{3}$. Let $f_1 = f - \frac{1}{3} = \frac{2}{39}$.

The largest unit fraction less than or equal to f_1 is $\frac{1}{20}$. $\frac{2}{39} - \frac{1}{20} = \frac{1}{780}$.

Since this is a unit fraction we stop and $\frac{5}{13} = \frac{1}{3} + \frac{1}{20} + \frac{1}{780}$.

Does this process always come to an end in a finite number of steps, and are the fractions produced always different – another Egyptian requirement?

Let f be $\frac{p}{q}$, and suppose that the largest unit fraction less than or equal to f be $\frac{1}{k}$.

Then we have $\frac{1}{k} \leq \frac{p}{q} < \frac{1}{k-1}$, so $f_1 = \frac{p}{q} - \frac{1}{k} = \frac{pk-q}{qk} \geq 0$

Also $\frac{1}{k-1} > \frac{p}{q} \Leftrightarrow \frac{q - p(k-1)}{(k-1)q} > 0 \Leftrightarrow \frac{p - (pk-q)}{(k-1)q} > 0$

Thus, since k > 1, $p > pk - q$, so the numerator of f_1 < numerator of f. (*)

Also the denominator of f_1 > denominator of f.

As the numerators of the successive f_1, f_2 . . . are always strictly decreasing integers, they must eventually reach 1, so the process must terminate in a unit fraction at some point.

To show that unit fractions are not repeated, we use a proof by contradiction. We will use f and f_1, but clearly the argument will be true for any successive fractions in the chain.

$f_1 = f - \frac{1}{k}$, as before, and suppose that the largest fraction less than or equal to f_1, is also $\frac{1}{k}$.

Then, since $-f_1 = \frac{p}{k} - \frac{1}{k} = \frac{pk-q}{kq} < \frac{p}{kq}$ [by (*) above]

so $f_1 - \frac{1}{k} < \frac{p}{kq} - \frac{1}{k} = \frac{f}{k} - \frac{1}{k} = \frac{1}{k}(f-1) < 0$.

But this is a contradiction since $\frac{1}{k}$ is less than or equal to f_1. Thus the assumption must be wrong, and the fraction cannot be repeated – all the fractions produced by this process must therefore be distinct.

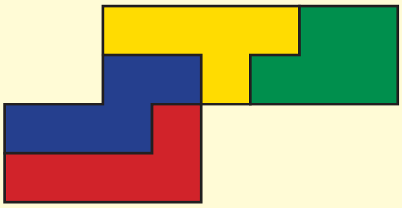
We do not know for certain how the Egyptians performed this decomposition. The method above, and the proof of its validity, were outlined by the great British mathematician J. J. Sylvester (1814-1897). After studying at Cambridge (though not being allowed to take his degree, in spite of coming second wrangler, because he was a Jew – a barrier later removed) he became a professor at University College, London, then in America and finally at Oxford, as Savilian Professor of Geometry.

Graham Hoare

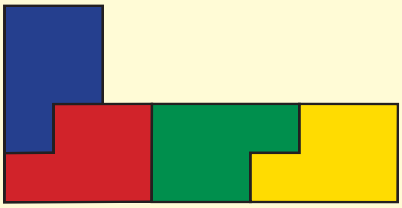
PENTOMINO DUPLICATION

In SYMmetry*plus* 5 we were introduced to 'pentomino triplication'. In a similar vein the process of 'duplication' can cause quite a few problems. A duplicated pentomino is, of course, twice as high and twice as wide as a normal pentomino, so covers twenty squares. It is very easy to construct all twelve of these from normal pentominoes; that isn't the problem.

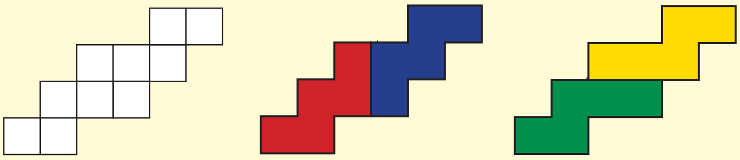
Problem 1 The diagram shows a double size pentomino made of four *different* pentominoes. How many others can you make? Some are not possible, some can be done in more than one way.



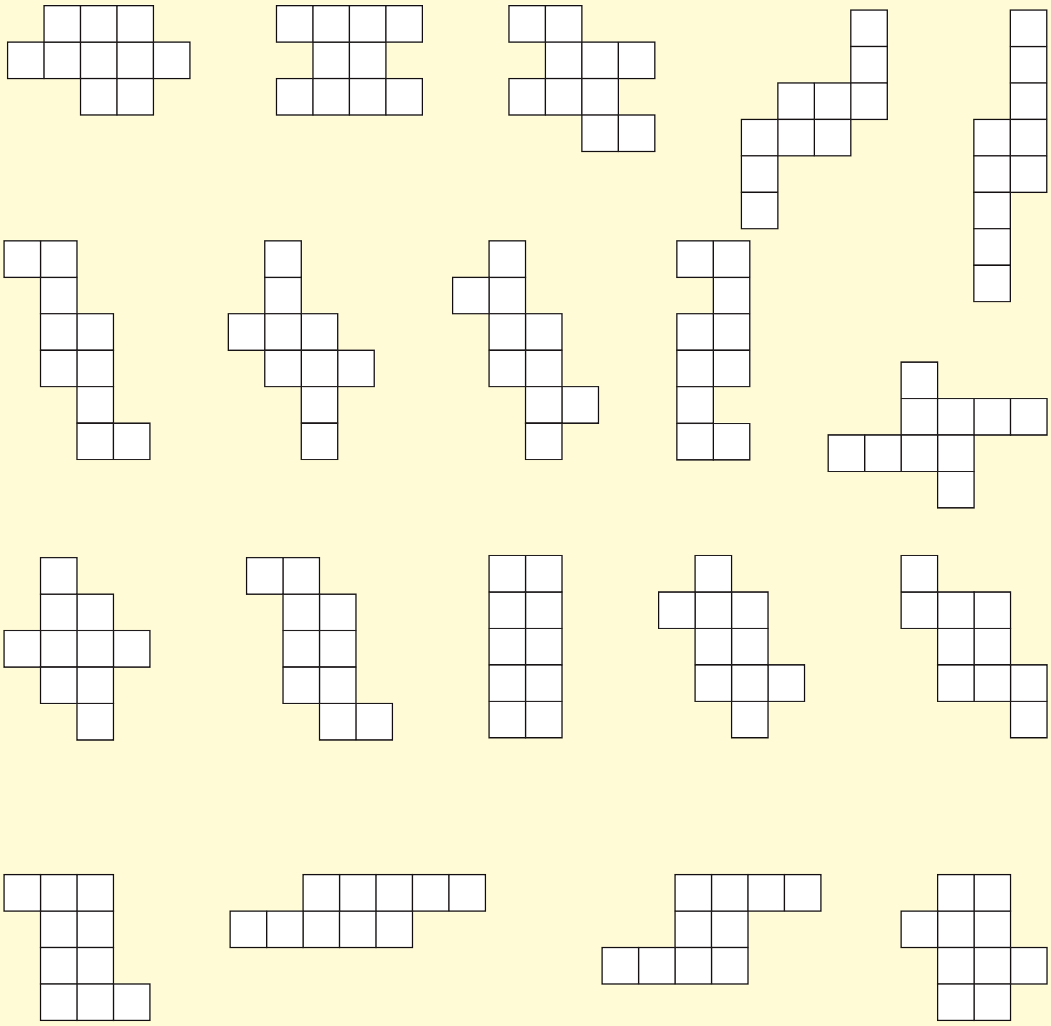
Problem 2 This diagram shows a duplicated pentomino made of four copies of the *same* pentomino. How many of these can you find. Again, some are not possible (but can be done with three copies plus another one) and some can be done in more than one way.



Problem 3



These diagrams show a shape made from two copies of the *same* pentomino in two different ways. Can you do the same with these patterns?



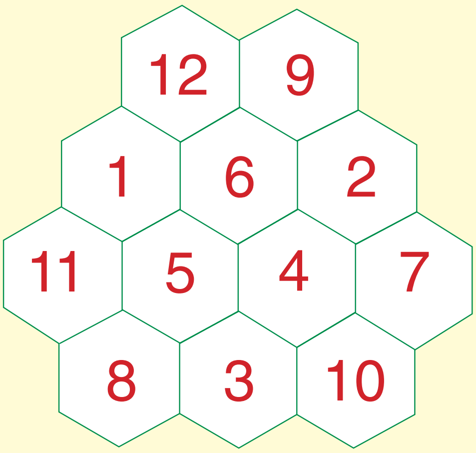
Brian Denison

SOME SOLUTIONS FROM LAST TIME

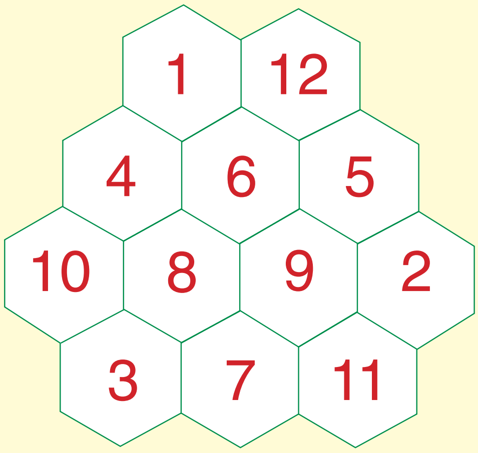
HEXAGON CIRCLES

The problem was to put the numbers 1 to 12 into the hexagons so that the total of the six numbers on the circumference of each of the three circles was the same. There are lots of ways of doing this, even after allowing for reflections and rotations.

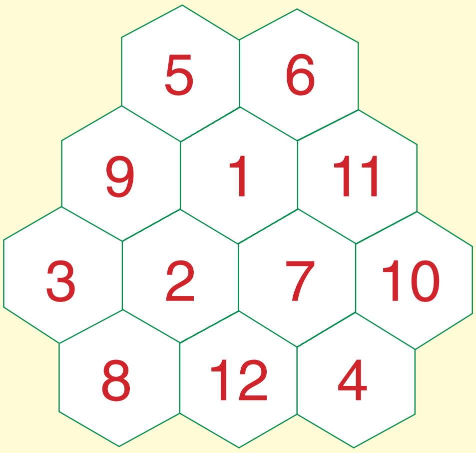
We are just showing four. The lowest total possible is 33, and the largest is 45.



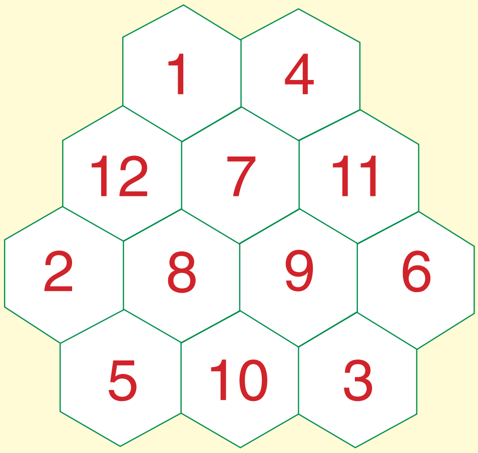
= 33



= 39



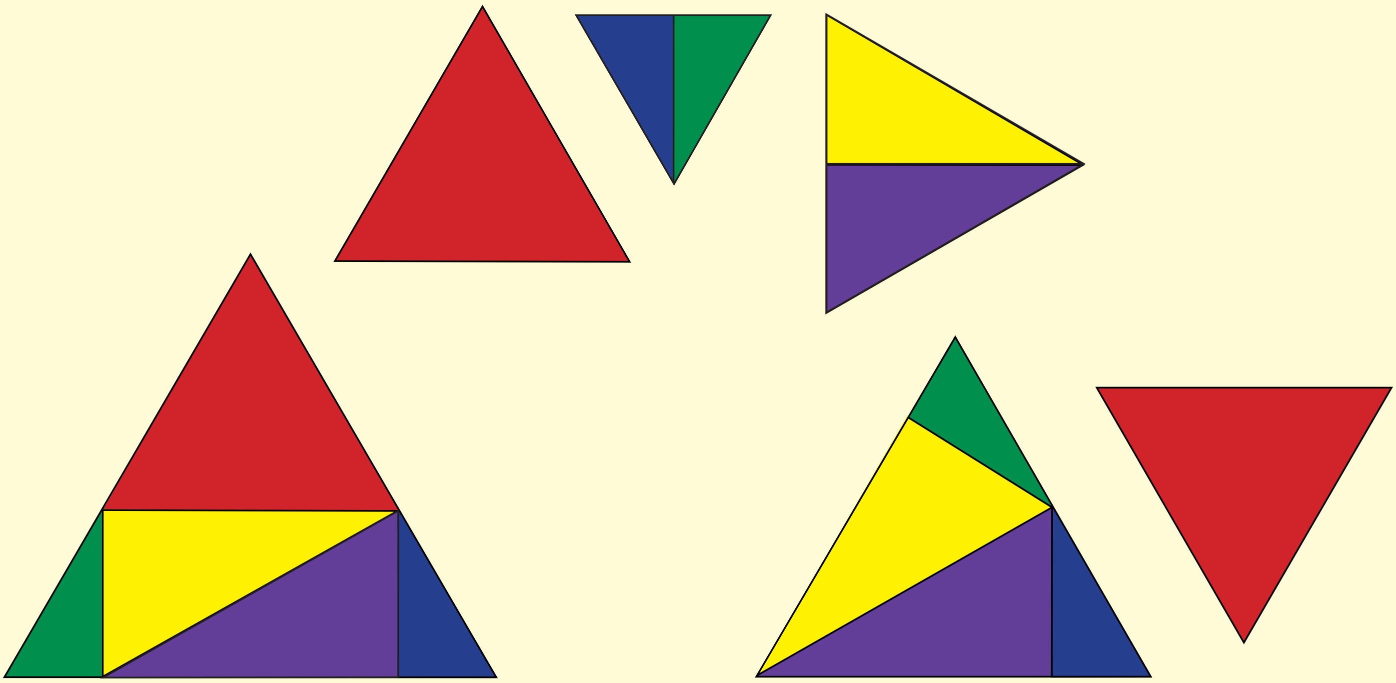
= 40



= 45

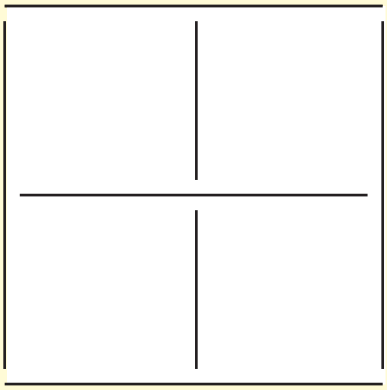
EQUILATERAL TRIANGLE

Dissect an equilateral triangle into five pieces which you can reassemble to give either 2 or 3 equilateral triangles. We gave a hint to think of a triangle of side 5 units.

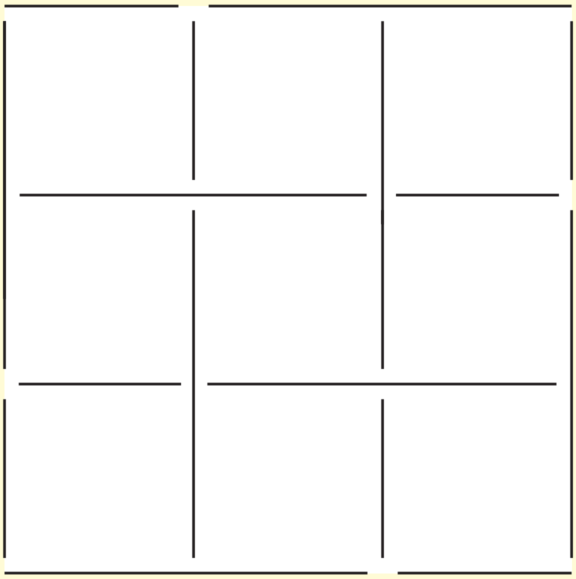


FRISKY LAMBS (AGAIN!)

With the original resources: four folds:



With double quantities: nine folds:



DOUBLETS

PIG	TEARS	BLACK	MINE	APE
PIT	SEARS	CLACK	MINT	APT
PAT	STARS	CLICK	MIST	OPT
SAT	STARE	CHICK	MOST	OAT
SAY	STALE	THICK	MOAT	OAR
STY	STILE	THINK	COAT	MAR
	SMILE	THINE	COAL	MAN
		WHINE		
		WHITE		

CHAIN LINKS

LINKS	*TOTAL*	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
3	6	5	1	3	2	4						
	7	5	2	1	4	3						
4	9	7	2	6	1	3	5	4				
	10	3	7	2	1	5	4	6				
	11	4	7	1	3	2	6	5				
5	11	9	2	5	4	6	1	7	3	8		
	13	4	9	1	3	8	2	5	6	7		
	13	9	4	1	8	3	2	5	6	7		
	14	5	9	2	3	4	7	1	6	8		
6	14	10	4	9	1	11	2	7	5	3	6	8
	14	10	4	8	2	11	1	7	6	3	5	9
	14	10	4	7	3	9	2	11	1	5	8	6
	15	6	9	2	4	10	1	11	3	5	7	8
	16	5	11	1	4	10	2	8	6	3	7	9
	16	11	5	1	10	4	2	8	6	3	7	9
	16	11	5	3	8	1	7	5	4	2	10	6
	17	6	11	2	4	8	5	3	9	1	7	10

CROSS-NUMBER SOLUTION FROM ISSUE 9

[1] 4	[2] 6	[3] 6	5	[4] 6	[5] 2	[6] 3	[7] 1	[8] 4
[9] 3	8	7	[10] 6	[11] 2	7	0	0	0
[12] 4	9	6	[13] 1	5	4	[14] 4	8	3
[15] 3	2	[16] 4	9	[17] 4	4	[18] 8	9	2
[19] 5	1	2	[20] 2	8	[21] 9	[22] 4	[23] 1	0
9	[24] 8	3	[25] 7	4	[26] 7	1	2	[27] 1
[28] 3	6	1	[29] 9	[30] 2	5	[31] 7	[32] 1	6
[33] 1	4	[34] 3	2	0	[35] 1	3	6	9
9	[36] 9	6	1	[37] 4	2	8	7	5

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NYOUT

This is a game which was probably played in Korea some three thousand years ago. It is one of a family collectively called 'cross & circle' games. The board consists of 29 points formed into a cross and circle. The points at the obvious points N, W, S, E, are larger, and the N point is marked Ch'ut, meaning 'exit'. The pieces are called *Mal,* representing horses. They are moved using four dice, known as *Pam-Nyout.* These are traditionally pieces of stick, flat and white on one side, curved and black on the other. This means that they can be modelled by tossing coins. The rules are:

1. There are two players: each has four horses
2. Each horse starts at N and moves anticlockwise to finish at N (an exact throw is not necessary).
3. Throws of the sticks (coins) count as follows: 1 flat (say, head) up scores 1, etc: but 0 up scores 5. If a score of 4 or 5 is achieved, the player has an extra throw; the scores so produced may be used for one horse or split between two or more horses. (Thus, for example, a player might throw 4, 5, 2; the possible moves are then: one horse 11; two horses 9+2, or 7+4, or 5+6; or three horses 5, 4 and 2)
4. If a horse ends its turn on W, S or E, it must next travel towards N via C.
5. If a horse ends on another of the player's own horses they join together and move as one. Three or four-composite horses are possible.
6. If a piece lands on a point occupied by an opponent, the opponent is sent back to the start if the piece landing is made up of at least as many horses as the occupying piece. A piece cannot land on a piece of higher order.