‘PIG’ AND OTHER TALES

A Book of Mathematical Readings

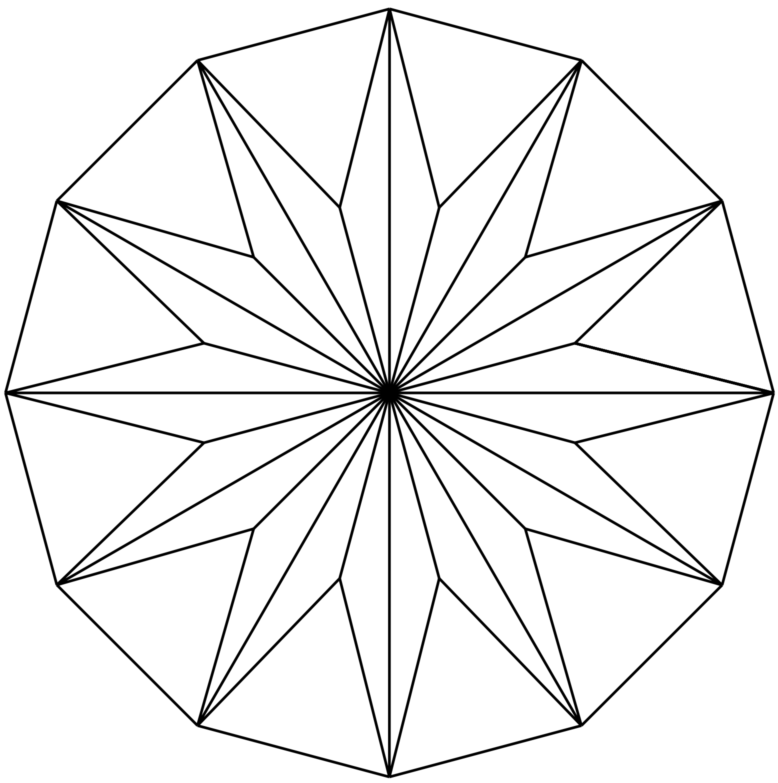
with

Questions and Specimen Answers

for

Students of A Level or Scottish Highers

Edited by Doug French and Charlie Stripp



The Mathematical Association

Introduction

The readings in this book have been chosen from past issues of the *Mathematical Gazette*, since it was felt that they could profitably be studied by A level and Scottish Higher Mathematics students. They cover a wide variety of topics, and whilst some are more difficult to understand than others, most should be accessible to the majority of such students.

The readings are accompanied by questions on their content. The questions are structured to help the student get the most out of each reading. The skill of reading and understanding mathematical or scientific articles is an important one, and is not one that is generally addressed. However, some A level boards are now introducing comprehension exercises into their examinations, for which these readings should be especially relevant. Some advice to students on reading mathematical articles is given on page (iii).

As well as questions, specimen answers are also included, which should help to stimulate discussion, as well as enabling students to check their solutions.

It is intended that the materials should be photocopied to allow their use to be as flexible as possible. Permission is given by the Mathematical Association to purchasers to make copies for use in their institutions.

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THE MATHEMATICAL ASSOCIATION

READINGS WITH QUESTIONS AND SPECIMEN ANSWERS FOR A LEVEL OR SCOTTISH HIGHERS STUDENTS

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Advice on Reading Mathematical Articles (iv)

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An analysis of the probabilities associated with a simple game which uses two dice.

2. **The Trouble with Asymptotes** 7
J Deft *Mathematical Gazette*, 72, October 1988
A comparison of three proposed methods for finding the equation of the oblique asymptotes to the graph of a rational function.

3. **The Glass Rod Problem** 12
G Haigh *Mathematical Gazette*, 65, March 1981
A neat pictorial solution to a probability problem.

4. **Bride's Chair Revisited** 17
R Webster *Mathematical Gazette*, 78, November 1994
An interesting property arising from the diagram used in Euclid's proof of Pythagoras' Theorem.

5. **Morley and More** 22
N Reed *Mathematical Gazette*, 65, March 1981
A proof of Morley's Theorem, which states that an equilateral triangle is formed by joining intersections of lines trisecting the angles of any triangle.

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Two related ways of displaying the complex roots of a quadratic on a graph.

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A fascinating dissection which provides an elegant way of finding the area of a regular dodecagon.

Advice on Reading Mathematical Articles

Reading a mathematical article is not like reading a novel. A novel is normally read straight through without any need to pause and think, or to re-read part in order to clarify some point. Some advice is given here about the sort of strategies which are helpful in reading effectively about mathematics.

- Always skim-read the article first to get a general idea of what it is about.
- Identify the key ideas involved and do not be distracted initially by details which are not central to the main purpose of the article.
- Be prepared to be persistent—some sections may need to be read several times to make sense of them.
- Constantly ask yourself questions about what you are reading.
- Have pencil, paper and a calculator available and use them to check calculations, fill in missing details, draw diagrams and clarify arguments.
- Aim to understand thoroughly—identify the words and symbols whose meaning is not clear.
- Think carefully about the logic of the arguments.
- If something fails to make sense after several attempts, read on—it may fall into place when the argument has been developed further.
- If it does not make sense today, try again tomorrow—starting afresh often results in you looking at previous problems from a different perspective.
- It is very difficult to produce a flawless article. Apart from misprints, authors may omit to state assumptions or definitions, or make clear which is which. If you cannot follow an argument, try to find a counter-example. If you think you have found one, check carefully to ensure it is genuine. This may reveal the significance of some remark that you previously overlooked. If the counter-example is genuine, check for unstated assumptions and so on. Such flaws seldom invalidate an article.

The purpose of the questions provided with each reading is to help the reader understand the article. They provide examples of the sort of questions that readers should ask themselves when reading a mathematical article.

Chapter 1 PIG

S Humphrey

Mathematical Gazette, **63**, December 1979

"Pig" is a game of dice which is simple yet exciting, which can be used for simple mental arithmetic practice and analysed with the help of probability theory.

The game

OBJECT. To be the first to score over 100.

NUMBER OF PLAYERS. Two or more.

NEEDED. Two dice, pen and paper to score.

HOW TO PLAY. On your turn you will roll both dice. If you throw a six with either die your score is 0 for that turn, and you pass the dice on to the next player. Otherwise add the numbers showing on the dice. You can now pass the dice on, or elect to throw again. If you pass the dice on without having thrown a six, your score is added to your previous scores. However, throwing a six at any time in a turn brings your turn to an end and your score is 0 for that turn.

Examples

	Score	Decision	Total for turn
(i)	4, 3	keep dice	
	2, 4	pass on	13
(ii)	1, 5	keep dice	
	3, 6		0
(iii)	2, 6		0
(iv)	4, 4	keep dice	
	2, 1	keep dice	
	3, 5	pass on	19

Observations

It does not take many turns to realise that the more shakes of the dice you try to have in any one turn, the greater chance there is of getting a six, and so scoring zero. On the other hand, if you restrict yourself to one shake of the dice only, you are less likely to get a six, which means you would expect to score, but not very highly.

What is the 'best number' of shakes to have in a particular turn?

Analysis

The probability of getting no sixes with one throw of the dice will be

$$\frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$$

With n throws of the dice, the probability of no sixes will be $(25/36)^n$.

Given that no sixes occur in a given throw, the expected total will be 6. After n similar throws the expected total will be $6n$.

Hence in the game itself the expected score after n consecutive throws could be written as

$$u_n = 6n\left(\frac{25}{36}\right)^n .$$

To one decimal place,

$$u_1 = 4.2,\ u_2 = 5.8,\ u_3 = 6.0,\ u_4 = 5.6.$$

Thus the best strategy appears to be to aim for a target of 3 consecutive throws at each turn before passing the dice on.

General case

Consider the game played with not 2, but k dice. Then with one throw of these dice the expected score will be $3nk\,(5/6)^{nk}$. With n throws of dice the expected score becomes $3nk\,(5/6)^{nk}$ which we denote by u_n. To find the value of n which maximises this, we reason as follows:

$$\begin{aligned} u_{n+1} > u_n &\Leftrightarrow 3(n+1)k\left(\frac{5}{6}\right)^{(n+1)k} > 3nk\left(\frac{5}{6}\right)^{nk} \\ &\Leftrightarrow (n+1)\left(\frac{5}{6}\right)^k > n \\ &\Leftrightarrow n\left\{1 - \left(\frac{5}{6}\right)^k\right\} < \left(\frac{5}{6}\right)^k \\ &\Leftrightarrow n < \frac{5^k}{6^k - 5^k}, \end{aligned}$$

and similarly with > or =.

Thus with $k = 1$, $u_5 = u_6$ are the maximum expectations. Otherwise the maximum occurs when n is the integer next greater than $5^k/(6^k - 5^k)$: that is $n = 3$ when $k = 2$, $n = 2$ when $k = 3$, and $n = 1$ for $k > 3$.

We could now lift the restriction of the original game of Pig to allow players in a particular turn to throw as many dice as they see fit!

S HUMPHREY

Questions on Chapter 1: PIG

1. Draw up a 6×6 table showing the possible totals when two fair dice are thrown.
2. Use this to explain why the probability of getting no sixes in one throw of the game is $\frac{25}{36}$.
3. Given that no sixes occur in a given throw, the possible total scores are from 2 to 10. Draw up a table of these scores and their related probabilities. Hence show that the expected total in one throw, given that no sixes have occurred, is 6.
4. Explain why, after n similar throws, the expected total is $6n$.
5. Explain how the expression $u_n = 6n\left(\frac{25}{36}\right)^n$ has been obtained.
6. Verify the values for u_1 to u_4 and calculate u_5.
7. Explain why the best strategy is 3 throws.
8. Consider the game played with one die only and show that $u_n = 3n\left(\frac{5}{6}\right)^n$. What is the best strategy if the game is played with one die?
9. Extend the argument for one and two dice to explain why, for k dice, the expected score is $3nk\left(\frac{5}{6}\right)^{nk}$.
10. Why, with $k = 1$, is u_5 exactly equal to u_6? In other words, without using a calculator or multiplying out, show that $15\left(\frac{5}{6}\right)^5 = 18\left(\frac{5}{6}\right)^6$.
11. Use the result in 9 to complete this table of expected scores.

Number of throws, n	Number of dice, k			
	1	2	3	4
1		4.2		
2		5.8		
3		6.0		
4		5.6		–
5			–	–
6		–	–	–
7		–	–	–

Note: Do not calculate those values which are blanked out as they are not essential to the argument.

12. Use this table to explain the penultimate paragraph of the article.
13. Evaluate $\dfrac{5^k}{6^k - 5^k}$ for $k = 1, 2, 3, 4$ and explain why this supports the previous results.
14. Explain carefully the steps in the argument which shows that if, and only if, $u_{n+1} > u_n$ then $n < \dfrac{5^k}{6^k - 5^k}$.

Answers to Chapter 1: PIG

1.

	6	7	8	9	10	11	12	
Score	5	6	7	8	9	10	11	TOTALS
on	4	5	6	7	8	9	10	WHEN
first	3	4	5	6	7	8	9	2 DICE
die	2	3	4	5	6	7	8	ARE
	1	2	3	4	5	6	7	THROWN
		1	2	3	4	5	6	

Score on second die

2. The shaded region shows the number of combinations which do not involve a six being thrown. There are 25 of these and a total of 36 combinations so:

$$P\,(\text{no sixes}) = \frac{25}{36}$$

3. *Given* no sixes thrown:

Possible scores	2	3	4	5	6	7	8	9	10
Probability	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{5}{25}$	$\frac{4}{25}$	$\frac{3}{25}$	$\frac{2}{25}$	$\frac{1}{25}$

Expected total:

$$= 2\times\tfrac{1}{25}+3\times\tfrac{2}{25}+4\times\tfrac{3}{25}+5\times\tfrac{4}{25}+6\times\tfrac{5}{25}+7\times\tfrac{4}{25}+8\times\tfrac{3}{25}+9\times\tfrac{2}{25}+10\times\tfrac{1}{25}$$

$$= \tfrac{1}{25}(2+6+12+20+30+28+24+18+10)$$

$$= \tfrac{1}{25}(150)$$

$$= 6$$

[Can also be explained by the symmetry of the table of probabilities]

4. If you throw n times without throwing a 6, the expected value is 6 for each throw, so the total expected is $6 \times n$, i.e. $6n$.

5. The probability of managing n throws without throwing a 6 is $[P\,(\text{no sixes})]^n$ i.e. $(\tfrac{25}{36})^n$.
As the expected score = score × probability of achieving no sixes, u_n, the expected score after n throws without a 6, is given by:

$$u_n = 6n\left(\frac{25}{36}\right)^n.$$

6.

$$u_1 = 6 \times \frac{25}{36} = 4.16 \text{ (3 sf)}$$

$$u_2 = 12 \times \left(\frac{25}{36}\right)^2 = 5.79 \text{ (3 sf)}$$

$$u_3 = 18 \times \left(\frac{25}{36}\right)^3 = 6.03 \text{ (3 sf)}$$

$$u_4 = 24 \times \left(\frac{25}{36}\right)^4 = 5.58 \text{ (3 sf)}$$

$$u_5 = 30 \times \left(\frac{25}{36}\right)^5 = 4.85 \text{ (3 sf).}$$

7. These expected scores increase steadily until $n = 3$ and then decrease again. The best strategy is that which gives the highest expected score so 3 throws is best.

8. Playing with *one* die:
   a) P(no six) = $\frac{5}{6}$.
   b) *Given* no 6 is thrown:

Possible outcomes	1	2	3	4	5
Probabilities	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

   c) Expected total after 1 throw: 3 [by symmetry or by Σ(score × probability)]
   d) Expected total score after n throws: 3n

   Probability of n throws without a 6: $\left(\frac{5}{6}\right)^n$.

   So, for one die: $u_n = 3n\left(\frac{5}{6}\right)^n$.

9. Playing with k dice:
   a) P(no sixes) = $\left(\frac{5}{6}\right)^k$.
   b) *Given* no 6 is thrown:
   Possible outcomes $k, k + 1, k + 2, \ldots, 5k + 1, 5k - 1, 5k$
   with symmetrical probabilities so:
   c) Expected total after 1 throw with k dice is: $\frac{k + 5k}{2} = 3k$.
   d) Expected total score after n throws: $3nk$.

   Probability of n throws without a 6: $\left(\frac{5}{6}\right)^{nk}$.

   So, for one die: $u_n = 3nk\left(\frac{5}{6}\right)^{nk}$.

10.
$$k = 1, \quad u_5 = 3 \times 5 \times \left(\frac{5}{6}\right)^5.$$
$$u_6 = 3 \times 6 \times \left(\frac{5}{6}\right)^6 = 18 \times \frac{5}{6} \times \left(\frac{5}{6}\right)^5 \quad \text{but} \quad 18 \times \frac{5}{6} = 15.$$
So
$$u_6 = 15 \times \left(\frac{5}{6}\right)^5.$$

11.

No. of throws	No. of dice (k)			
(n)	1	2	3	4
1	2.5	4.2	5.2	5.8
2	4.2	5.8	6.0	5.6
3	5.2	6.0	5.2	4.0
4	5.8	5.6	4.0	–
5	6.0	4.8	–	–
6	6.0	–	–	–
7	5.9	–	–	–

Table showing expected scores (1 dp).

12. The maximum expected scores can be seen clearly in this table. They occur in the first column (one die) with 5 *or* 6 throws, in the second column (2 dice) with 3 throws, in the third column (3 dice) with 2 throws and in the fourth column (4 dice) on the first throw. As the number of dice increases, the best number of throws has decreased steadily, so the trend seen in this table suggests that the first throw gives the best expected score for any number of dice above 3 (i.e. $k > 3$).

13.

k	$\dfrac{5^k}{6^k - 5^k}$	Next integer greater than this
1	5	5 or 6
2	2.27 (2 dp)	3
3	1.37 (2 dp)	2
4	0.93 (2 dp)	1

Now the theory suggested that the best number of throws, n, is the next integer greater than $\dfrac{5^k}{6^k - 5^k}$ for each value of k. These values are shown to the right of the values of $\dfrac{5^k}{6^k - 5^k}$ shown above and are the same as those arrived at previously.

14.

$$u_{n+1} = 3(n+1)k\left(\frac{5}{6}\right)^{(n+1)k} \qquad u_n = 3nk\left(\frac{5}{6}\right)^{nk}$$

So $u_{n+1} > u_n \Leftrightarrow 3(n+1)k\left(\frac{5}{6}\right)^{(n+1)k} > 3nk\left(\frac{5}{6}\right)^{nk}$

$$\Leftrightarrow \quad (n+1)\left(\frac{5}{6}\right)^{nk}\left(\frac{5}{6}\right)^{k} > n\left(\frac{5}{6}\right)^{nk} \qquad \text{(dividing by } 3k\text{)}$$

$$\Leftrightarrow \quad (n+1)\left(\frac{5}{6}\right)^{k} > n \qquad \text{(dividing by } \left(\tfrac{5}{6}\right)^{nk}\text{)}$$

$$\Leftrightarrow \quad \left(\frac{5}{6}\right)^{k} > n - n\left(\frac{5}{6}\right)^{k} \qquad \text{(collecting } n \text{ terms)}$$

$$\Leftrightarrow \quad \left(\frac{5}{6}\right)^{k} > n\left(1 - \left(\frac{5}{6}\right)^{k}\right) \qquad \text{(factorising)}$$

$$\Leftrightarrow \quad 5^k > n(6^k - 5^k) \qquad \text{(multiplying by } 6^k\text{)}$$

$$\Leftrightarrow \quad n < \frac{5^k}{6^k - 5^k} \qquad \text{as required.}$$

Chapter 2: THE TROUBLE WITH ASYMPTOTES

J Deft

Mathematical Gazette, **72,** October 1988

Considerable staffroom discussion was generated recently by a question which asked for a sketch of

$$y = \frac{(x+3)(x+1)}{(x-2)}$$

Clearly there is an asymptote $x = 2$, and another oblique asymptote whose equation has to be found.

"Easy," said Aloysius. "When x is very large, the most significant term on the top line is x^2, and on the bottom is x. The fraction then becomes (near enough) x^2/x, and so the asymptote is $y = x$."

"I don't do it that way," said Balthazar. "Obviously

$$y = \frac{x^2 + 4x + 3}{x - 2},$$

and if we divide the top and bottom lines by x we get

$$y = \frac{x + 4 + 3/x}{1 - 2/x}.$$

When x is large, $3/x$ and $2/x$ tend to zero and we are left with the asymptote $y = x + 4$."

"I don't get either of those answers," said Cordelia. "I agree with

$$y = \frac{x^2 + 4x + 3}{x - 2},$$

but then by algebraic long division I get

$$y = x + 6 + \frac{15}{x-2},$$

and when x tends to infinity this gives an asymptote $y = x + 6$."

As far as the sketch is concerned, of course, it really doesn't matter –the graph disappears from the top right-hand corner of the paper at an angle of approximately 45°–but clearly only one of these answers can be right. Some simple numerical work shows that the correct solution is in fact Cordelia's, and it is then fairly easy to see that each of the other methods depends on the erroneous assumption that the limit or tendency of a quotient is equal to the quotient of the limits. Nevertheless, the discussion drew quite forcefully to our attention the unreliability of the "quick methods" that we had been using happily for many years.

JOHN DEFT

Questions on Chapter 2: THE TROUBLE WITH ASYMPTOTES

1. Where does the curve cut the x and y axes?

2. Why is there an asymptote at $x = 2$?

3. What happens as x gets closer to 2 from above and below?

4. Illustrate Aloysius's argument numerically and explain how it *does* show that y is very large and positive when x is very large and positive.

5. Explain in the same way what happens to y when x is very large and negative.

6. Sketch the curve on the basis of the information established so far and compare with a curve plotted on a graphical calculator or a graph plotter.

7. What is meant by an *oblique* asymptote?

8. Show that the line $y = x$ cuts the curve when $x = -0.5$ and, assuming the two branches of the curve are on opposite sides of the asymptote, explain with reference to your sketch why $y = x$ cannot be the oblique asymptote.

9. Explain Balthazar's argument.

10. Show that the line $y = x + 4$ cuts the curve when $y = -5.5$ and explain why it cannot be the oblique asymptote.

11. Explain how Cordelia showed that $\dfrac{x^2 + 4x + 3}{x - 2} = x + 6 + \dfrac{15}{x - 2}$.

12. Why does this show that $y = x + 6$ is the oblique asymptote?

13. What happens if you try to find where $y = x + 6$ cuts the curve?

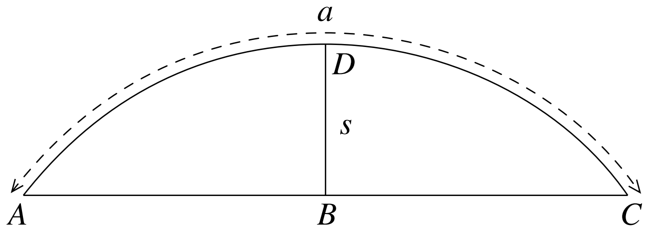
14. What does it mean by the *erroneous assumption* that the *limit of a quotient is equal to the quotient of the limits*?

Chapter 8: BENDING THE SHEET

H Martyn Cundy

Mathematical Gazette, 72, October 1988

An engineer friend put to me the following practical problem, which turned out to be quite an interesting one. He wants to bend a sheet of metal, of width a, on a cylindrical mandrel so that a 'cloche' is formed whose central height will be s. What should be the diameter of the cylinder? He found this not to be quite as easy as it sounds! Sixth formers might find it an interesting investigation.



If 2θ is the angle subtended by the sheet at the centre of the cylinder, whose diameter d is to be found, we need $0 \leqslant \theta \leqslant \pi/2$ and we have the two equations

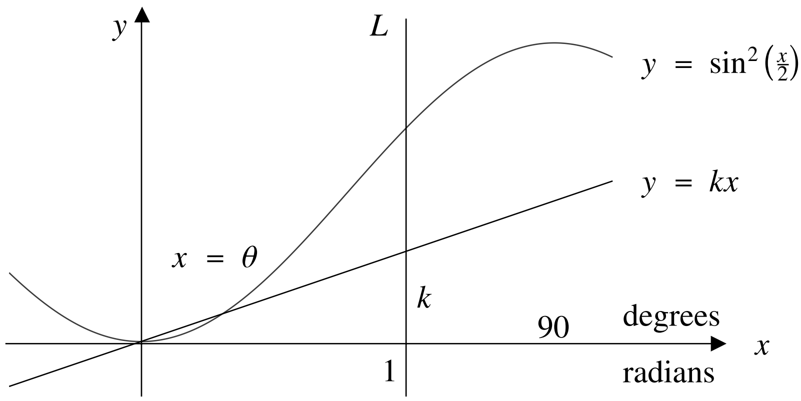
$$\theta = \frac{a}{d} \qquad \text{and} \qquad \sin^2\left(\frac{\theta}{2}\right) = \frac{s}{d}$$

from which we need to eliminate θ. If $s/a = k$ we have to find the θ for which

$$\sin^2\left(\frac{\theta}{2}\right) = k\theta.$$

For the process under consideration, k is likely to lie in the range (0, 0.25). Three methods of proceeding suggest themselves.

1. Graphical If we draw the graph of $y = \sin^2(x/2)$ (where x can be measured for convenience in degrees) and a line L given by $x = 1$ radian (why here?), then we can graduate L with values of k on a suitable scale, so that a straight edge laid joining the origin to the k-mark on L will cut the graph between 0° and 90° where $x = \theta$. If y is the ordinate at the point of intersection, the diameter required will be given as s/y.



This method does not give great accuracy, and is ill-conditioned for small k. Maybe we could find a series in k which would approximate to d?

2. Series We have

$$k\theta = \sin^2\left(\frac{\theta}{2}\right) = \tfrac{1}{2}(1 - \cos\theta)$$

so that

$$2k\theta = 1 - \cos\theta = \frac{\theta^2}{2} - \frac{\theta^4}{24} + \frac{\theta^6}{720} - \ldots$$

and, by repeated substitutions on the right,

$$\begin{aligned}\theta &= 4k + \frac{\theta^3}{12} - \frac{\theta^5}{360} + \ldots \\ &= 4k + \frac{\left(4k + \frac{\theta^3}{12} - \frac{\theta^5}{360} + \ldots\right)^3}{12} - \frac{\left(4k + \frac{\theta^3}{12} - \frac{\theta^5}{360} + \ldots\right)^5}{360} + \ldots \\ &\vdots \\ &= 4k + \frac{16k^3}{3} + \frac{832k^5}{45} + \ldots .\end{aligned}$$

Finally, we can use the series expansion of $(1 + x)^{-1}$ to obtain

$$d = \frac{a}{\theta} \approx \frac{a}{36k}\left(9 - 12k^2 - 25.6k^4 - 100k^6\right).$$

an unexpectedly neat series which is accurate to 6 significant figures.

3. Computing Finally, it is easy to devise an iterative procedure for θ which converges fairly rapidly in the range considered. The obvious iteration to solve

$$\sin^2\left(\frac{\theta}{2}\right) = k\theta \qquad (\text{or } 2k\theta = 1 - \cos\theta)$$

is

$$\theta_{n+1} = \frac{1}{k}\sin^2\left(\frac{\theta_n}{2}\right)$$

but this does not converge. However

$$\theta_{n+1} = \cos^{-1}\left(1 - 2k\theta_n\right)$$

behaves very well and is easily programmed on a pocket calculator.

H MARTYN CUNDY

Questions on Chapter 8: BENDING THE SHEET

1. If the sheet of metal is 1.5m wide and the 'cloche' is to be 30cm in height, show that $k = 0.2$ where $k = \frac{s}{a}$ with s and a as in the diagram.

2. Draw a diagram showing the angle of 2θ subtended by the sheet at the centre of the cylinder.

3. Why do we need $0 \leqslant \theta \leqslant \frac{\pi}{2}$?

4. Explain how the equation $\theta = \frac{a}{d}$ has been found.

5. Calling the centre of the circle O on your diagram, and using triangle AOD, show that $AD = d \sin\frac{\theta}{2}$.
Hence, using triangle ABD, show how the equation $\sin^2\left(\frac{\theta}{2}\right) = \frac{s}{d}$ has been found.

6. By eliminating d between the two equations considered in questions 4 and 5, show that $\sin^2\left(\frac{\theta}{2}\right) = k\theta$, where $k = \frac{s}{a}$.

7. At the point of intersection where $x = \theta$, explain why $y = k\theta = \frac{ka}{d}$.
Use the definition of y to show that the diameter is $\frac{s}{y}$.

8. Explain how the iterative formula $\theta_{n+1} = \cos^{-1}(1 - 2k\theta_n)$ has been obtained.
Draw the graph of $y = x$ and $y = \cos^{-1}(1 - 2kx)$ to show how the iteration converges.

9. Compare methods 1 and 3 by finding a value for θ when $k = 0.2$.

10. Try the Newton-Raphson formula as an alternative method.

11. Explain how the series for $2k\theta$ has been obtained from the series for $\cos\theta$.

12. By dividing through by $2k\theta$ and rearranging, show how the series for θ in terms of k and higher powers of θ has been obtained.

13. Explain what is meant by the *repeated substitutions on the right*, which have been used to obtain θ as a series in k.

14. What is the series expansion of $(1 + x)^{-1}$?
How has it been used to determine the series for d?

Chapter 11: MATHEMATICS AND THE MOTION OF THE HUMAN BODY

T Roper

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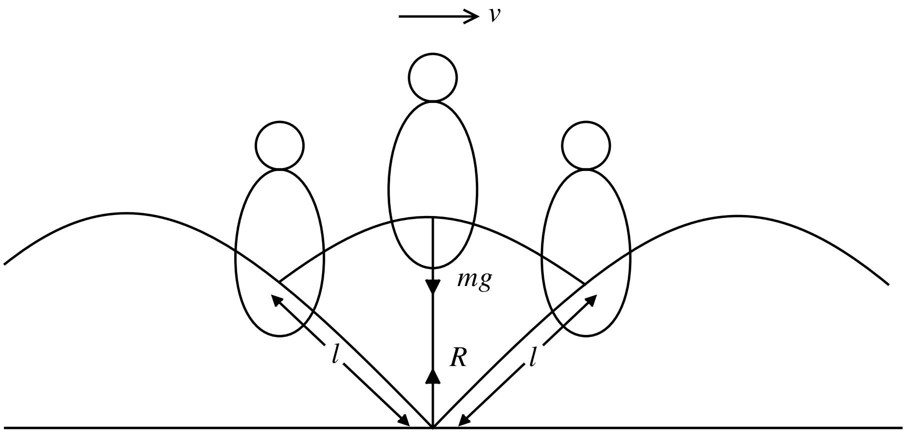
1. Walking—how fast can we walk?

The crucial point about walking is that there must always be contact with the ground, if contact is lost then running is taking place. How fast can we walk before the gait changes to running? Other questions stem from this, for example what features of the race-walkers' curious gait enable them to walk faster than ordinary mortals? Why do we take shorter steps when walking faster?

In a talk given to a group of P.G.C.E. students, Professor R. McNeil Alexander of the Biology Department, University of Leeds suggested the following model:

> Suppose that the Centre of Gravity (C. G.) of the body moves in a series of circular arcs whose radius is the length of the pivoting leg, l, and whose centre is on the ground at the foot, whilst the body moves forward in a straight line with speed v.



Using Newton's Second Law when the body is in vertical alignment

$$mg - R = m\frac{v^2}{l} \Rightarrow R = mg - \frac{mv^2}{l}.$$

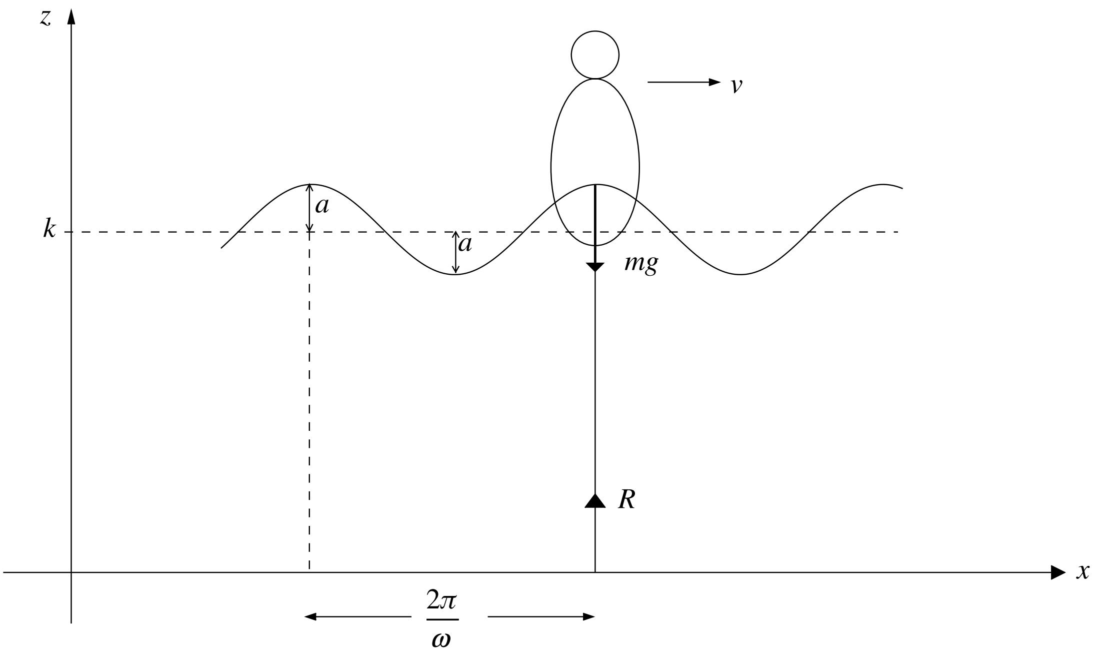
But for walking we must maintain contact with the ground and so

$$R > 0 \Rightarrow lg > v^2.$$

Now $l \approx 0.9$ m and $g \approx 10\,\text{m/s}^2$, hence $9 > v^2$, giving a maximum walking speed of about 3 m/s; a not unreasonable result. The inequality $lg > v^2$ shows the dependency of v upon both the length of the pivoting leg and the acceleration due to gravity. Consider first the dependency upon length of leg: clearly the shorter the leg is the lower the speed of walking. Hence when parent and toddler walk hand-in-hand, the parent walks but the toddler often runs. With regard to gravity, on the moon the acceleration due to gravity is about one-sixth that on earth, hence on the moon $v \approx 1.2\,\text{m/s}$. This is a very slow walking speed: hence the astronauts who "walked" on the moon were seen to "bounce" rather than walk.

The model is crude and if walking were actually to take place in this way it would be extremely painful and literally bone-shattering. Nevertheless it is extremely effective in explaining observable features and provides a reasonable solution to the initial problem. It is a very good model of the situation.

However, photographs, graphs and figures in [4], a volume of research on the human gait conducted around the turn of the century for the Prussian army, suggests that the path of the C.G. could be modelled by a smooth curve very like the sine curve in appearance although no effort is made to fit any such curve. Suppose then that we model the path of the C.G. by the curve $z = k + a \sin \omega x$.



Maxima represent points of contact of a single foot with the ground and so the distance between adjacent maxima is equal to the stride length, s.

Thus, $s = 2\pi / \omega$ and this gives an interpretation to ω in terms of one of the characteristics of the situation. Using Newton's Second Law at a maximum point we obtain

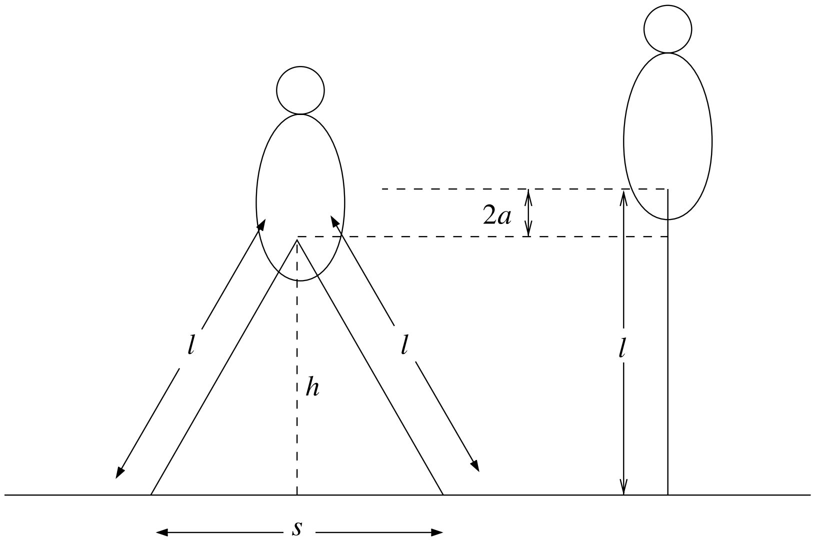
$$mg - R = m \frac{v^2}{|\rho|} \Rightarrow R = mg - mav^2\omega^2$$

where ρ is the radius of curvature ($= -1/a\omega^2$) at the point in question. Again for walking we require $R > 0$ and so we have

$$\frac{g}{a\omega^2} > v^2.$$

Our two inequalities now explain the gait of race-walkers. For the leg in contact with the ground there is a marked backwards rotation of the hip whilst the opposite side of the body "drops" down. The backwards rotation of the hip effectively increases the length of the pivoting leg and so from the first inequality, v is increased. The dropping of the opposite side of the body reduces the variation in the vertical displacement of the C.G.; hence a is reduced, not only helping to reduce the expenditure of energy, but (from the second inequality) providing an increase in v. (See [5] for diagrams and fuller explanations.)

We have already related ω to the stride length, and a can be similarly related if we make the assumption that maxima occur when the bcdy is in vertical alignment with one foot on the ground, and minima occur when the weight is carried equally by both feet.



From the diagram

$$2a = l - \sqrt{l^2 - \frac{s^2}{4}}$$

and so substituting into the second inequality and using $s = 2\pi/\omega$ gives

$$v^2 < \frac{g}{a\omega^2} \Rightarrow v^2 < \frac{gs^2}{2\pi^2\left(l - \sqrt{l^2 - \frac{s^2}{4}}\right)} = \frac{gs^2}{\pi^2\left(2l - \sqrt{(4l^2 - s^2)}\right)}$$

$$= f(s), \text{ say.}$$

Now f is a decreasing function of s for $0 < s < 2l$. Hence our observable experience that stride length shortens as the speed increases is borne out.

How good is the second model? Calculations based upon figures in [4] give good agreement with the first model for the maximum speed v, and fitting a curve of the form described above gives good agreement between ω and s, the stride length of the subject of the experiments in [4]. However the above inequality gives comparatively low maximum walking speeds: this is because an even distribution of weight is not achieved as described above and so a is rather less than $l - \sqrt{\left(l^2 - \frac{1}{4}s^2\right)}$.

References

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4. W. Braune and O. Fischer, *The human gait*, (Translated by P. Maquet and R. Furlong. Springer-Verlag, 1987 : original German edition published by B. G. Teubour 1895-1904.)
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Questions on Chapter 11: MATHEMATICS AND THE MOTION OF THE HUMAN BODY

1. What is the maximum height of the centre of gravity above the ground in the first model?

2. What is the significance of each of the three terms in the equation $mg - R = \frac{mv^2}{l}$ and why are they related in this way?

3. Why is $R > 0$ and why does it follow that $lg > v^2$?

4. Convert 3m/s into miles per hour, given that one mile is approximately 1.6km.

5. Explain how the maximum walking speed of 1.2m/s on the moon has been calculated.

6. Why is the stride length s given by $s = \frac{2\pi}{\omega}$ on the second model?

ρ is described as the radius of curvature at the point on the curve where the centre of gravity is highest. It is the radius of the circle which fits the curve best at that point. We need not be concerned here with how it has been obtained, or the significance of the negative sign.

7. How has the inequality $\frac{g}{a\omega^2} > v^2$ been obtained from the third figure?

8. Explain the details of the two diagrams showing the maximum and minimum positions.

9. How has the equation $2a = l - \sqrt{l^2 - \frac{s^2}{4}}$ been obtained?

10. Show that if we take $g \approx 10$, $\pi^2 \approx 10$ and $l = 0.5$ then the expression for $f(s)$ becomes $f(s) = \frac{s^2}{1 - \sqrt{1 - s^2}}$.

11. Evaluate this at $s = 1$, $s = 0.5$, $s = 0.1$, $s = 0.01$ and $s = 0.001$. What does this suggest about the value of the function as s tends to 0? Why, in practice, must s be less than 1 in this case? Use the values to sketch a graph of $f(s)$.

12. Remembering that $v^2 < f(s)$, how does the graph show that 'stride length shortens as speed increases'?

Answers to Chapter 11: MATHEMATICS AND THE MOTION OF THE HUMAN BODY

1. Maximum height is l, the length of the pivoting leg.

2. mg is the weight of the person.
   R is the normal reaction force from contact with the ground.
   $mg - R$ is the resultant downward force on the person when the leg is vertical.
   $\frac{v^2}{l}$ is the acceleration towards the centre of the circle (foot on the ground) over which the centre of gravity is travelling.
   $\frac{mv^2}{l}$ is the mass × acceleration.

   By Newton's Second Law $F = ma \Rightarrow mg - R = \frac{mv^2}{l}$.

3. If $R = 0$ then the foot would not be in frim contact with the ground, so $R > 0$ for firm contact.

   $$mg - R = \frac{mv^2}{l} \Rightarrow R = mg - \frac{mv^2}{l}$$

   $$R > 0 \Rightarrow mg - \frac{mv^2}{l} > 0 \Rightarrow gl > v^2.$$

4. $$3 \text{ ms}^{-1} = \frac{3 \times 60 \times 60}{1000} \text{ kmh}^{-1} = 10.8 \text{ kmh}^{-1}$$

   $$10.8 \text{ km} = \frac{10.8}{1.6} \text{ miles} = 6.75 \text{ miles}$$

   Thus $3 \text{ ms}^{-1} = 6.75$ mph.

5. $gl > v^2$.
   On the moon $g \approx \frac{1}{6} \times 10 \text{ ms}^{-2}$.
   Assuming $l = 0.9$, then $gl > v^2$ gives $0.9 \times \frac{1}{6} \times 10 > v^2 \Rightarrow v^2 < 1.5$. Taking the square root gives a maximum walking speed of about 1.2 ms^{-1}.

6. One stride allows the pivot point to complete one period of the sine wave, $a \sin \omega x$.
   Period = $\frac{2\pi}{\omega}$ so $s = \frac{2\pi}{\omega}$.

7. $|\rho|$ is the radius of the circle that best fits the curve when the leg is vertical. So assuming the pivot point is travelling with speed v in a circle of radius $|\rho|$,

   $$mg - R = \frac{mv^2}{|\rho|} \text{ and } |\rho| = \frac{1}{a\omega^2}$$

   $$\Rightarrow mg - R = \frac{mv^2}{1/(a\omega^2)} = mav^2\omega^2$$

   $$\Rightarrow R = mg - mav^2\omega^2.$$

   $$R > 0 \Rightarrow mg - mav^2\omega^2 > 0 \Rightarrow mg > mav^2\omega^2 \Rightarrow \frac{g}{a\omega^2} > v^2$$

8. The diagram illustrating the minimum position shows the two legs forming an isosceles triangle, consistent with the assumption that, at the minimum position, the weight is carried equally by both feet. At this point the centre of gravity is at a minimum point on the sine wave. At the maximum position, with one leg vertical, and bearing all the weight, the centre of gravity is at a maximum point on the sine wave. The sine wave has amplitude a, so the maximum position is a height $2a$ above the minimum position.

9. $2a$ is the difference in height between the lowest and highest pivot points.
   By Pythagoras' Theorem

$$l^2 = h^2 + \left(\frac{s}{2}\right)^2 \Rightarrow h^2 = l^2 - \frac{s^2}{4} \Rightarrow h = \sqrt{l^2 - \frac{s^2}{4}}$$

$$2a = l - h \Rightarrow 2a = l - \sqrt{l^2 - \frac{s^2}{4}}.$$

10.
$$f(s) = \frac{gs^2}{\pi^2\left(2l - \sqrt{4l^2 - s^2}\right)}; \text{ with } g \approx 10,\ \pi^2 \approx 10,\ l = 0.5$$

$$f(s) = \frac{10s^2}{10\left(2 \times 0.5 - \sqrt{4 \times 0.5^2 - s^2}\right)} \Rightarrow f(s) = \frac{s^2}{1 - \sqrt{1 - s^2}}.$$

11.

s	1	0.5	0.1	0.01	0.001
$f(s)$	1	1.87	1.99	2.00	2.00

So as $s \to 0$, $f(s) \to 2$.
s = stride length. If the leg length, l, is 0.5 m, then the maximum possible stride is 1m so $s < 1$ in this case.

12. $v^2 < f(s)$.
   If, at constant stride length, v is steadily increased, then running will start when $v^2 = f(s)$. To increase v further, $f(s)$ must increase.
   From question 11, if $f(s)$ increases then s decreases.
   So, if speed increases, the maximum stride length shortens.