12: $\frac{1}{2}+\frac{1}{4}=\frac{2}{6}$? $=\frac{1}{6}$? or what?

The problem is the same as you encounter when doing the sum

1 football player + 1 football team.

How many is it, and how many of *what*?

What we can do is calculate, say,

5 football players +3 football players or 2 teams + 4 teams,

namely, add things of the same type. So let us see if we can convert the first sum above to something where we add things of the same *type*. We could do so by replacing the type 'team' with the type 'player'. But this is not just about changing words – we must retain the given contents (that is, the meaning of the word 'team' as a certain number of football-players). Because we know that 1 team means 11 players, we can rewrite the above

1 player + 1 football team as a manageable task, namely:

1 player + 11 players (12 players)

Note that doing it the other way round, i.e. replacing 'player' by 'team' would be less convenient:

1 player is $\frac{1}{11}$ of a team, so converting the whole thing into teams would give

$\frac{1}{11}$ team + 1 team $=1\frac{1}{11}$ teams (the same as 12 players, but less neat).

If the referee plays for the other team – and he always does - then they are a $1\frac{1}{11}$ team!

Now we apply the same method to $\frac{1}{2}$ + $\frac{1}{4}$ one half + one quarter. Either halves or quarters –

which is more convenient? Should we state how many quarters equal one half or vice versa? The

former, of course: 1 *half* is 2 *quarters*.

So, the $\frac{1}{2}$ becomes $2\times\frac{1}{4}$, i.e. $\frac{2}{4}$, and the above task becomes $\frac{2}{4}+\frac{1}{4}$, meaning

2 quarters + 1 quarter $=$ 3 quarters, i.e. $\frac{3}{4}$

In the above sum, 1 *half* + 1 *quarter*, the different types were associated with the different denominators (dividers), so the problem was solved by a transformation that led to identical denominators. Let us now do this for the more general case:

$\frac{2}{3}+\frac{5}{12}$ (where we have more than one of each type, namely: ***2*** thirds and ***5*** twelfths).

Again, we want the same denominators under both, either 3 or 12. Which shall we choose? We must remember that the values of the fractions must remain unchanged, so, if we change the denominator, we must also change the numerator (the divided). If we choose the 3 as the common denominator, we must change the 12 in the second term into a 3, that is, make the denominator 4 times smaller. To keep the value of the fraction unchanged, we must then also make the 5 on top (the numerator) 4 times smaller, resulting in 5/4 as the numerator. This is inconvenient because it creates another fraction above the dividing line.

Instead, we can choose the second denominator, the 12, as the common one. We leave the $\frac{5}{12}$ as it is and change the first denominator, the 3, to 12. This we do by increasing the 3, by multiplying it by 4. Then, to protect the value of this fraction, we must also make the numerator (the 2) 4 times bigger, this time giving a manageable whole number, 8. *Note that this resulted from choosing the larger of the two denominators as the common one*.

And so, the above $\frac{2}{3}+\frac{5}{12}$ now becomes $\frac{8}{12}+\frac{5}{12}$, 8 twelfths +5 twelfths =13 twelfths, i.e. $\frac{13}{12}$.

But what about $\frac{2}{3}+\frac{5}{4}$?

To get the first (smaller) denominator (the 3) to be the same as the second denominator 4, the 3 has to be multiplied by 1.33…, and the same must then be done to the 2 above it, which results in a messy 2.66…! So we now need a common denominator which is neither 3 nor 4. In principle, we could choose anything to put equally at the bottom of both fractions. The only problem is that we need to adjust the numerators (to keep the values of the given fractions unchanged) and as we found, we may only do this by multiplying the numerators by whole numbers. Form this follows that the denominators, too, can be changed only by multiplying by whole numbers.

The task becomes: By what whole number do we multiply the 3 (the denominator), and by what (different) whole number do we multiply the 4 (the other denominator) so that in both cases we get the same result (namely, the same common, denominator)? The nice trick for this is to multiply the first denominator (3) by the second denominator (4) and the second denominator (4) by the first (3) ! (always, of course, multiplying the numerator by the same as the denominator).

So this is what was done:

$$\frac{2\times 4}{3\times 4}+\frac{5\times 3}{4\times 3},\ \text{which equals}\ \frac{8}{12}+\frac{15}{12}\ =\ \frac{8+15}{12}\ =\ \frac{23}{12}$$

Generalizing:

$$\frac{n}{a}+\frac{m}{b}\ =\ \frac{n\times b}{a\times b}+\frac{m\times a}{b\times a}\ =\ \frac{n\times b+m\times a}{a\times b}$$

A wealthy merchant left his three sons 17 camels – the oldest was to get one half of them, the next son one third and the youngest son to get one ninth. How many did they each get ? Left to themselves they'd have been still arguing about it, so wisely they went to an uncle who worked like this. First he lent them a camel so they had 18 camels in all. Then,

$$\frac{1}{2}x18=9;\ \ \frac{1}{3}x18=6\ \ \text{and}\ \ \frac{1}{9}x18=2;$$

this accounted for 9+6+2=17 camels and that meant the uncle could reclaim the one he had lent! But what is $\frac{1}{2},\frac{1}{3}$ and $\frac{1}{9}$ of 17 ?